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CALCULATION OF A STEADY-STATE THERMAL REGIME IN ELECTRON-BEAM AUTOCRUCIBLE MELTING IN THE CASE OF CIRCULAR SCANNING WITH A BEAM

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We suggest a mathematical model of a steady-state thermal regime in electron-beam autocrucible melting. We carry out a numerical investigation of the dependence of the thermal parameters of melting on the radius of circular scanning by the beam, the focal spot radius, and on the mean rate of mixing of the melt. The results of calculations are compared with experimental data.

The essence of the process of electron-beam autocrucible melting (EBAM) consists of the melting out of an alloy from a lumpy or loose charge loaded into a smelting reservoir (autocrucible) with subsequent production of cast articles from the melt. In view of the recent tendency to increase the fraction of cast articles instead of those produced by deformation or machining, EBAM has found an increasingly important place in casting not only from refractory and chemically active metals, but also from nickel-, iron-, and cobalt-based alloys.

In technological investigations of electron-beam melting and EBAM one usually makes use of a certain averaged (integral) temperature based on the mean rate of metal evaporation [1] or measured experimentally at fixed points on the heated surface, for example, at the center of the bath [2] or in the middle of its radius in a smelting vessel [3]. Such an approach, however, is suitable only when a uniform distribution of the electron-beam heating power is ensured, and it is a very rough approximation of the actual conditions of EBAM, which is usually carried out by focused electron beam scanning the heated surface. Certain features of the nonstationary technique of electron-beam heating were taken into account in [4] when constructing physical models of EBAM that aided in obtaining analytical dependences for the heating temperature of metals at the focal spot. However, only the use of mathematical models opens up possibilities for comprehensive consideration of the majority of factors that influence the process of melting and makes it possible to determine not only the temperature of the metal at any point in the autocrucible but also the volume of the autocrucible bath and the superheating of the melt in it [5-7].

A special feature of the EBAM process in the regime of holding at a prescribed constant power of heating is the attainment of a stationary distribution of the metal temperature in the autocrucible and stabilization of the volume of the liquid bath when the incoming energy of electron-beam heating is completely removed by a cooling system and by radiation and evaporation heat losses from the surface heated.

Below we suggest a mathematical model of a stationary thermal regime in an autocrucible. The model is constructed using the approach of [5-7] based on heat conduction theory with the introduction, for the liquid phase, of the coefficient of effective heat conduction, which imitates forced convective heat transfer in a melt. Unlike [5-7], we take into account the scanning of the heated surface with an electron beam, dependence of the effective heat conduction coefficient on the liquid bath diameter at a prescribed, mean over the bath, speed of melt motion, as well as the temperature dependence of the heat conduction coefficient of the solid metal.

We will describe the process considered and aspects of the mathematical model design. The autocrucible has a cylindrical shape, and its side ($r = a$) and bottom ($z = 0$) surfaces are cooled with water (the cooling agent can be other than water); heating is provided by a focused electron beam that continually scans the surface of the

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metal $z = l$ according to a program (circle, spiral, intersecting lines, etc.). The character of the heat source in simulation of electron-beam heating is dictated by the magnitude of the accelerating voltage of the electron gun, which influences the depth of the penetration of electrons into the metal [8]. In the majority of cases, EBAM is carried out using electron guns of the axial type with an accelerating voltage of 40–50 kV [4]. Calculations by Shonland's formula [8] show that for metals at such accelerating voltages the electron penetration depth does not exceed 10^{-5} m; therefore, the relative error arising in calculation of the metal temperature by means of the mathematical model involving a surface source of electron-beam heating (to whose consideration we will restrict ourselves in the present work) does not exceed 0.01% compared to the case when volume heat absorption is taken into account.

An important factor that substantially influences the temperature field and makes it possible to increase the discharge of liquid metal, simultaneously decreasing its heating, is electromagnetic mixing of the melt, which causes intense heat exchange in the entire volume of the bath. We will use the very popular [4-9] hypothesis that forced convective heat transfer in a melt can be simulated by means of the coefficient of effective heat conduction λ_e, which exceeds $\tilde{k}$ times the coefficient of molecular heat conduction λ_L. To find the value of $\tilde{k}$, we shall use a formula obtained as a result of experimental investigations of turbulent heat transfer for the case of forced convection [10]:

$$\tilde{k} = \begin{cases} 0.45\,(\mathrm{Pr\,Re})^{0.438} & \text{for}\quad \mathrm{Pr\,Re} \le 8600\,, \\ 1.35 \cdot 10^{-6}\,(\mathrm{Pr\,Re})^{1.84} & \text{for}\quad \mathrm{Pr\,Re} > 8600\,. \end{cases} \tag{1}$$

Taking into account the fact that $\mathrm{PrRe} = 2vr_*C_{VL}/\lambda_L$, we see that the quantity $\tilde{k}$ depends on the heat capacity of a unit volume of liquid metal C_{VL}, the bath-mean speed of melt motion v, and the mean radius of the bath r_*. We note that in [4] the value $\tilde{k} = 10$ was used in calculations for rather intense mixing of the metal.

In the case of electron-beam scanning over a circle with its center on the crucible axis $r = 0$ or simply of axisymmetric heating by a fixed beam, assuming that the effective heat conduction coefficient λ_e = const in the entire region of the liquid phase, determination of the developed temperature field of the metal $T(r, z)$ and of the melting isotherm (phase interface) $r = R(z)$ requires solution of the following axisymmetric stationary problem of the Stefan type [5, 6]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda(T)\, r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\lambda(T)\frac{\partial T}{\partial z}\right) = 0\,, \quad (r, z) \in \Omega_s\,;$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0\,, \quad (r, z) \in \Omega_L\,;$$

$$\frac{\partial T}{\partial r} = 0\,, \quad r = 0\,; \quad \lambda(T)\frac{\partial T}{\partial r} + \alpha_1 T = \alpha_1 T_w\,, \quad r = a\,; \tag{2}$$

$$\lambda(T)\frac{\partial T}{\partial z} - \alpha_2 T = -\alpha_2 T_w\,, \quad z = 0\,; \quad \lambda(T)\frac{\partial T}{\partial z} = q(r) - f(T)\,, \quad z = l\,;$$

$$T = T_m\,, \quad r = R(z)\,; \quad \lambda_e \frac{\partial T}{\partial n}\bigg|_{r=R(z)-0} = \lambda(T)\frac{\partial T}{\partial n}\bigg|_{r=R(z)+0}\,.$$

Here $\Omega_s \cup \Omega_L = \{(r, z) : 0 < r < a;\ 0 < z < l\}$; $\Omega_s = \{(r, z) : T(r, z) < T_m\}$; $\Omega_L = \{(r, z) : T(r, z) > T_m\}$ are the regions of the solid and liquid phases, respectively; the linear boundary conditions at $r = a$ and $z = 0$ describe convective heat removal from the cooled surfaces of the crucible with the heat transfer coefficients α_1 and α_2, and the nonlinear boundary condition at $z = l$ is the absorption of the electron-beam energy of density $q(r)$ and heat losses due to radiation and evaporation from the heated surface, where $f(T) = \varepsilon\sigma T^4 + \eta(T - T_m)Q_{ev}(T)$. The last conditions in system (2) express the equality of the temperature at the phase interface $r = R(z)$ to the melting temperature T_m and the equality of heat fluxes from the liquid and solid phases in a steady state of the system.

We will represent the function $Q_{ev}(T)$, which denotes the evaporation heat flux density, in the form $Q_{ev}(T) = c_1 \exp(-c_2/T)$, where the constants c_1 and c_2 can be found both from the data of experimental investigations and using the Clapeyron–Clausius law written for a thin gas layer adjoining the evaporation surface [11, 12].

Approximate solution of problem (2) by the method of finite differences is complicated, since the position of the phase interface and the value of the coefficient λ_e are unknown. At the same time, the use of a time-dependent technique [13] requires solution of the corresponding nonstationary Stefan problem, which is extremely time-consuming. Therefore, we will reduce problem (2) to the equivalent nonlinear integral equation of Hammerstein [5, 6]. For this purpose, we shall use the Kirchhoff transformation

$$\psi = \int_0^{T-T_w} \lambda(s)\, ds$$

and also approximate the dependence of the thermal conductivity coefficient on temperature by a piecewise continuous function:

$$\lambda(T) = \lambda_i, \quad T_{i-1} \le T < T_i, \quad i = \overline{1, K} \quad (T_K = T_m, \; T_0 = T_w); \quad \lambda(T) = \tilde{k}\lambda_L, \quad T > T_m.$$

Excluding $T(r, z)$ from Eq. (2), we obtain a simpler boundary-value problem for the new unknown function $\psi(r, z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < l; \quad \frac{\partial \psi}{\partial r} + h_1\psi = 0, \quad r = a;$$
$$\frac{\partial \psi}{\partial r} = 0, \quad r = 0; \quad \frac{\partial \psi}{\partial z} - h_2\psi = 0, \quad z = 0; \quad \frac{\partial \psi}{\partial z} = q(r) - f(T(\psi)), \quad z = l \tag{3}$$

and the condition for determining the phase interface $r = R(z)$:

$$\psi(R(z), z) = \lambda_1(T_1 - T_w) + \sum_{i=2}^{K-1} \lambda_i (T_i - T_{i-1}) + \lambda_K (T_m - T_{K-1}). \tag{4}$$

Using the corresponding Green's function $G(\rho, \zeta; r, z)$, we reduce the solution of boundary-value problem (3) to a nonlinear integral equation of the Hammerstein type for the function $\varphi(r) = \psi(r, l)$ [5]. In order to find its approximate solution, we approximate $\varphi(r)$ by a piecewise continuous function, having denoted the mean-integral value of $\varphi(r)$ by φ_i over the interval (r_{i-1}, r_i), where $r_i = ia/M$ are the points of partitioning of the interval $[0, a]$ into M parts. After application of the zonal method (the simplest of the projection methods [14]), we obtain the following system of nonlinear equations for determining φ_j, $j = \overline{1, M}$:

$$\varphi_j = \varphi_{0j} - \sum_{i=1}^{M} G_{ij} f(T(\varphi_i)), \quad j = \overline{1, M}. \tag{5}$$

where the constants φ_{0j}, G_{ij} are found by integration of the known functions.

After determination of φ_j, $j = \overline{1, M}$, we can represent the distribution of the modified temperature in the form of the series

$$\psi(r, z) = \frac{2}{a^2} \sum_{n=1}^{\infty} A_n(z) J_0(\gamma_n r). \tag{6}$$

where

$$A_n(z) = \left[\gamma_n \int_0^a rq(r) J_0(\gamma_n r)\, dr/P_n - \sum_{i=1}^{M} f(T(\varphi_i)) A_{ni}\right] g_n(z);$$

$$A_{ni} = [r_i J_1(\gamma_n r_i) - r_{i-1} J_1(\gamma_n r_{i-1})]/P_n;$$

$$P_n = (\gamma_n^2 + h_1^2)\,[\gamma_n + h_2 - (\gamma_n - h_2)\exp(-2\gamma_n l)]\, J_n^2(\gamma_n a);$$

$$g_n(z) = (\gamma_n + h_2)\exp(-\gamma_n(l - z)) + (\gamma_n - h_2)\exp(-\gamma_n(l + z));$$

$0 < \gamma_1 < \gamma_2 < \ldots$ are roots of the equation $h_1 J_0(\gamma a) - \gamma J_1(\gamma a) = 0$.

However, it is impossible to solve system (5) without finding the constant $\bar{K}$, which was used for determining the coupling between the functions T and ψ. Returning to formula (1) for $\bar{K}$, we see that at the prescribed value of the mean speed of motion of the melt v the determination of $\bar{K}$ is reduced to finding the mean-integral value of the bath radius

$$r_* = \int_{z_0}^{l} R(z)\, dz/(l - z_0) \qquad (7)$$

over its depth $H = l - z_0$, which is also to be determined. This equation must be added to system (5) of the main equations of the problem.

After solving system (5), (7) using Eq. (6), it is possible to determine the unknown stationary temperature field $T(r, z)$, and from condition (4) to determine the position of the phase interface $r = R(z)$. The liquid bath volume, the mean-integral overheating of the melt over the entire volume of the bath, and the overheating of the melt on its surface are calculated by the formulas

$$W = \pi \int_{z_0}^{l} R^2(z)\, dz, \quad \Delta T = \frac{2\pi}{W} \int_{z_0}^{l} \int_0^{R(z)} T(r, z)\, r dr dz - T_m,$$

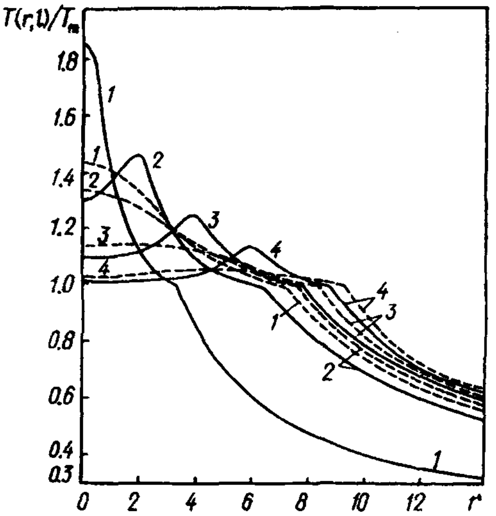
$$\Delta T_l = \frac{2}{R^2(l)} \int_0^{R(l)} T(r, l)\, r dr - T_m.$$

Now, we shall determine $q(r)$, which is the stationary distribution of the power density on the heating surface in the case of scanning of a circle of radius $0 < R < a$ with an electron beam. Let O_1 be the point of beam focusing in the plane $z = l$ at distance $|OO_1| = R$ from the point O lying in the same plane on the crucible axis. Let us consider the point $M(\beta)$ located at distance r from point O and at distance $y(\beta)$ from point O_1, where $\beta = \angle O_1OM(\beta)$. The value of the power density at point $M(\beta)$ is equal to $q_0 \exp[-k_r y^2(\beta)]$, where $y^2(\beta) = R^2 + r^2 - 2rR\cos\beta$, $0 \le \beta \le \pi$ (from the theorem of cosines). The density of the energy absorbed at each point that is located at distance r from point O for the time of passage of one full circle of radius R by the electron beam (for the period of rotation $T_{sc} = 2\pi R/v_{sc}$) is determined by the integral

$$E(r) = 2q_0 \int_0^{T_{sc}/2} \exp\left\{-k_r[r^2 + R^2 - 2Rr\cos(v_{sc}t/R)]\right\} dt =$$

$$= \frac{2Rq_0}{v_{sc}} \exp[-k_r(r^2 + R^2)] \int_0^{\pi} \exp(2k_r rR\cos\beta)\, d\beta.$$

As $q(r)$, it is natural to take the mean value of the density of energy absorbed per unit time



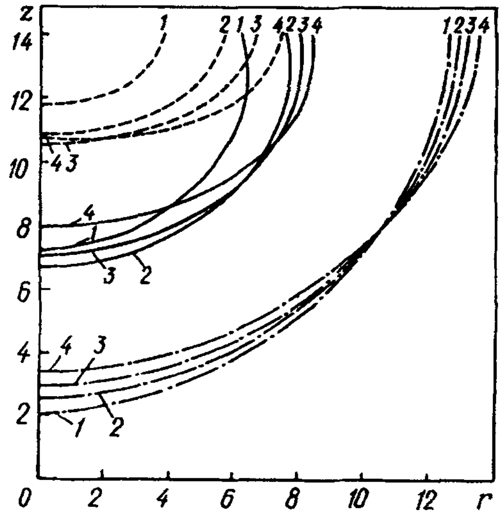


Fig. 1. Distribution of dimensionless temperature of niobium over heated surface at $P = 133$ kW; $v = 0.3$ m/sec; $b = 0.01$ m (solid curves) and $b = 0.04$ m (dashed curves) for different values of radius of circular scanning with a beam. R, m: 1) $R = 0$, 2) 0.02, 3) 0.04, 4) 0.06. r, cm.

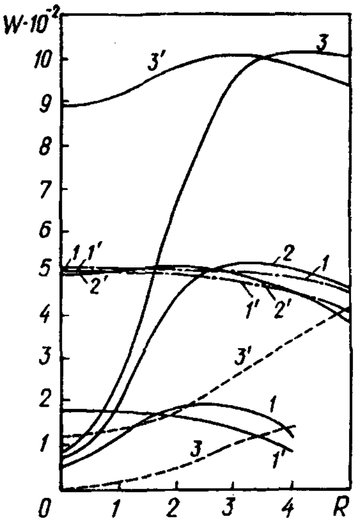
Fig. 2. Cross-sections of surfaces of separation of liquid and solid phases $r = R(z)$ obtained for niobium for $v = 0.3$ m/sec (solid curves), and $v = 0.005$ m/sec (dashed curves) and without allowance for losses by radiation and evaporation from heated surface (dashed-dotted curves) at $P = 133$ kW; $b = 0.03$ m and different values of the scanning radius R, m: 1) $R = 0$, 2) 0.03, 3) 0.04, 4) 0.05. r, z, cm.

$$q(r) = \frac{E(r)}{T_{sc}} = \frac{q_0}{\pi} \exp\left[-k_r\left(r^2 + R^2\right)\right] \int_0^{\pi} \exp\left(2k_r r R \cos\beta\right) s\beta . \tag{8}$$

Using the mathematical model developed, we carried out calculations of steady-state thermal regimes in electron-beam melting of niobium in an autocrucible of diameter 280 mm ($a = 0.14$ m) for a level of the metal in the crucible $l = 0.14$ m and electron beam power $P_0 = 190$ kW. According to [4], the efficiency of electron-beam heating is $\eta = 0.7$, therefore the magnitude of the power absorbed by the metal is taken to be equal to $P = \eta P_0 = 133$ kW.

For calculations we assumed [4, 15] that $T_m = 2740$ K; $C_{VL} = 0.2772 \cdot 10^7$ J/(m$^3 \cdot$K); $T_w = 300$ K; $\lambda_L = 56.2716$ W/(m·K); $\alpha_1 = \alpha_2 = 400$ W/(m$^2 \cdot$K); $\varepsilon = 0.4$; $c_1 = 0.31102 \cdot 10^{18}$; $c_2 = 93{,}868.526$ (we obtained the values of c_1 and c_2 on the basis of the experimental data given in [5]). The piecewise continuous dependence of the heat conduction coefficient on temperature was constructed using the data of [15] on the thermal conductivity of niobium. We considered the cases of melting using a system of electromagnetic mixing, when, according to [4], $v = 0.3$ m/sec, and without forced mixing of the melt ($v = 0.005$ m/sec). The corresponding values of $\bar{K}$ were determined in the process of problem solution and were equal to: a) from 8.34 to 13.11 at $b = 0.01$ m and from 11.79 to 12.88 for $b = 0.04$ m at $v = 0.3$ m/sec for values of the scanning radius within the range of from 0 to 6 cm, i.e., they were close to the value $\bar{K} = 10$ [4]; and b) from 1.44 to 2.00 at $v = 0.005$ m/sec, $b = 0.03$ m for the scanning radii 0–5 cm.

To determine the error arising in solving nonlinear boundary-value problem (4) by an approximate method, we calculated the magnitude of overall losses of energy from the surfaces $z = l$, $z = 0$, and $r = a$; for an exact solution of the considered stationary problem this magnitude should coincide with the magnitude of the power P absorbed by the metal. The results of calculations made using the data for niobium and the above-given parameters of the autocrucible for $b = 0.03$ m show that the error of determining the temperature on the surfaces $z = l$, $z = 0$, and r



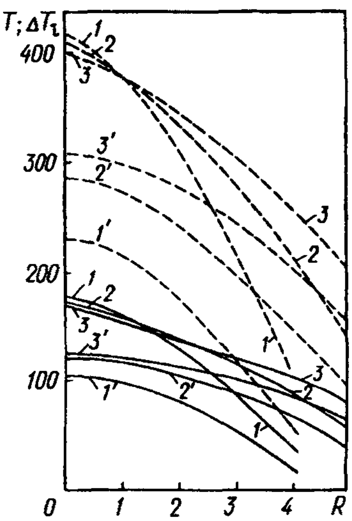


Fig. 3. Dependences of volume of niobium liquid bath $W \cdot 10^{-2}$, cm^3, on the radius of circular scanning R, cm, obtained for $v = 0.3$ m/sec (solid curves), and $v = 0.005$ m/sec (dashed curves) and without allowance for losses by radiation and evaporation from heated surface (dashed-dotted curves) at $b = 0.01$ m (numbering 1-3) $b = 0.04$ m (1'-3') and different values of absorbed power P, kW: 1, 1') $P = 65$; 2, 2') 100; 3, 3') 133.

Fig. 4. Dependences of mean-integral overheating of niobium melt throughout entire volume of bath ΔT, K (solid curves) and on its surface ΔT_l, K (dashed curves) on scanning radius R, cm, for $b = 0.01$ m (numbering 1-3) and $b = 0.04$ m (1'-3') at different values of absorbed power P, kW: 1, 1') $P = 65$; 2, 2') 100, 3, 3') 133.

$= a$, which is calculated with respect to the magnitude of absorbed power, is equal to 1.36% at $R = 0$ and up to 0.165% at $R = 0.03$ m.

The behavior of the temperature curves $T(r, l)$ at different values of R (Fig. 1) reproduces the behavior of the corresponding functions $q(r)$ determined in the form of Eq. (8). Increasing the scanning and focal spot radii leads to a more uniform distribution of the absorbed energy density over the heating surface and, consequently, to a decrease in the temperature gradient, as well as, naturally, to an increase in the radius of the bath on its surface $z = l$. The points of attainment of maximum temperature recede from the center $r = 0$ of the bath surface with an increase in R; in this case the temperature $T(0, l)$ decreases, approaching the value of the melting temperature T_m; therefore, in the case of an extreme increase in the scanning radius ($R > 0.06$ m) the formation of a unmelted zone in the central part of the bath surface is possible. The temperature curves have an inflection point in passage through the value equal to the melting temperature; this is due to the difference between the heat conduction coefficients of the melt and solid metal.

The depth of the bath increases with an increase in R (Fig. 2), reaching a maximum at a certain value of the scanning radius, and it decreases with a further increase in R. Due to heat losses from the heated surface, the depth and radius of the bath decrease approximately by a factor of two. In the case of melting with electromagnetic mixing, the bath has the greatest diameter not on the surface $z = l$, but somewhat lower; thus, the phase interface has a small bend to the side of the axis $r = 0$, which imparts an ellipsoidal shape to the bath observed in practice. The presence of such a bend is explained by the overestimation of thermal losses over the densities of the energy absorbed at the points of the intersection of the melting isotherm surface with the surface $z = l$, as a result of which the temperature derivative along the axial coordinate z is negative near the surface $z = l$. In the two other cases, it is correspondingly positive ($v = 0.005$ m/sec) and equal to zero ($f = 0$). We note that in the case of a linear problem ($f = 0$) the phase interface $r = R(z)$ is independent of the effective heat conduction coefficient.

The graphs given in Figs. 3 and 4 show that with an increase in the scanning radius, the volume of the bath increases, and the overheating decreases, which is quite explainable in view of the more uniform distribution of the absorbed energy over the heating surface with an increase in R. However, in the case of melting with electromagnetic mixing, an extreme increase in the scanning radius leads to a decrease in the volume, and at each value of "b" the function $W(R)$ has a point of maximum attainment. Comparing the $W(R)$ curves obtained for v = 0.005 m/sec and v = 0.3 m/sec, we see that the use of electromagnetic mixing allows one to increase the volume 7.5–20 times at R = 0 (depending on the value of b) and, for example, 2.8– 3.5 times at R = 5 cm. Applying circular scanning of the beam, we can increase the volume of the bath (compared to axisymmetric heating by a stationary beam) by a factor of twelve at b = 1 cm and by 15% at b = 4 cm. Due to heat losses from the heated surface the volume of the liquid bath decreases 2.5–4 times at absorbed power P = 65–133 kW. The overheating of the melt on the surface of the bath exceeds the mean-integral overheating throughout the entire volume of the bath by more than two times (Fig. 4).

To check the correspondence of the mathematical model developed to actual thermal regimes of EBAM, we compared the results of calculation with known experimental data [4]. In experimental melting of niobium in an autocrucible 280 mm in diameter with application of electromagnetic mixing at a maximum electron-beam power P_0 = 190 kW the mass of the discharged liquid metal varied from 8.4 kg to 8.7 kg, which, being converted to volume, amounts to 976.7 cm^3–1011.6 cm^3. This band of values is intersected by the $W(R)$ curves that correspond to the value P = 133 kW (Fig. 3) for a scanning radius of from 3 to 5 cm.

According to [4], in the case of EBAM without forced mixing the depth of the bath of zirconium in a 250 mm-diameter crucible is 24–28 mm. According to our calculations carried out for zirconium at a = 0.125 m; l = 0.1 m; α_1 = 400 W/($m^2 \cdot K$); α_2 = 50 W/($m^2 \cdot K$); v = 0.005 m/sec; b = 0.03 m; R = 0.35 m and values of λ and C_{VL} taken from [4], the depth of the bath varies from 20 to 27.2 mm with a change in the absorbed power of from P = 52.5 kW to P = 105 kW.

Thus, the results obtained by calculation find quite satisfactory experimental confirmation; therefore, the mathemtical model developed can be used in technological investigations of EBAM.

NOTATION

r, z, radial and axial coordinates; a, b, radii of crucible and focal spot; l, height of metal in crucible; R, scanning radius; z_0, point of intersection of phase interface with the $0z$ axis; $T(r, z)$, temperature field of metal; T_m, melting temperature; T_w, temperature of water in cooling system; $r = R(z)$, equation of the surface of separation of liquid and solid phases; λ_L, coefficient of molecular heat conduction of liquid metal; λ_e, coefficient of effective heat conduction; C_{VL}, heat capacity of unit volume of liquid metal; v, bath-mean speed of melt motion; r_*, mean-integral radius of bath; P_0, power of electron-beam heating; P, power absorbed by metal; $q(r)$, surface distribution of power density P; q_0, power density at point of beam focusing; α_1, α_2, coefficients of heat transfer from side and bottom surfaces of crucible; ε, emissivity of heated surface of metal; σ, Stefan–Boltzmann constant; $\eta(T - T_m)$, Heaviside unit function; $h_i = \alpha_i/\lambda_{si}$; λ_{si}, mean values of heat conduction coefficient on cooled surfaces, i = 1, 2; $k_r = 2b^{-2}$, coefficient of concentration of heat source in radial direction; v_{sc}, speed of beam scanning; W, volume of liquid bath; ΔT, ΔT_l, mean-integral overheating of melt over entire volume of bath and on its surface; J_0, J_1, Bessel functions of the first kind of zero and first order.

REFERENCES

1. A. S. Kalugin, Electron-Beam Melting of Metals [in Russian], Moscow (1980).
2. Yu. A. Kurapov, Processes of Vacuum Refining of Metals in Electron-Beam Melting [in Russian], Kiev (1984).
3. V. A. Shcherbina and V. D. Dovbnya, in: Progressive Means of Melting for Cast Shapes [in Russian], Kiev (1978), pp. 100-107.
4. S. V. Ladokhin and Yu. V. Kornyushin, Electron-Beam Autocrucible Melting of Metals and Alloys [in Russian], Kiev (1988).

5. A. A. Berezovskii and V. D. Dovbnya, in: Nonlinear Boundary-Value Problems [in Russian], Kiev (1980), pp. 41-57.
6. T. A. Andreeva, A. A. Berezovskii, and V. D. Dovbnya, in: Mathematical Models in Power Engineering [in Russian], Kiev (1988), pp. 106-122.
7. A. A. Berezovsky, V. D. Dovbnia, and Yu. V. Zhernovyi, in: Condensed Matter: Science and Industry Abstracts, Information and Participants, Ukrainian-French Symposium, Lviv (1993), p. 133.
8. N. N. Rykalin, I. V. Zuev, and A. A. Uglov, Fundamentals of Electron-Beam Treatment of Materials [in Russian], Moscow (1978).
9. B. I. Medovar (ed.), Thermal Processes in Electroslag Remelting [in Russian], Kiev (1978).
10. L. A. Volkhonskii, Vacuum Arc Furnaces [in Russian], Moscow (1985).
11. A. A. Uglov, I. Yu. Smurov, and A. M. Lashin, Teplofiz., Vysok. Temp., 27, No. 1, 87-93 (1989).
12. R. M. Cherniga and I. G. Odnorozhenko, Promysh. Teplotekh., 13, No. 4, 51-59 (1991).
13. N. M. Belyaev and A. A. Ryadno, Mathematical Methods of Heat Conduction [in Russian], Kiev (1993).
14. G. I. Marchuk and V. I. Agoshkov, Introduction into Projection-Grid Methods [in Russian], Moscow (1981).
15. A. N. Zelikman, B. G. Korshunov, A. V. Elyutin, and A. M. Zakharov, Niobium and Tantalum [in Russian], Moscow (1990).